GEOMETRY AND ITERATION OF DIANALYTIC TRANSFORMATION

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ABSTRACT – This paper is a continuation of the paper [5] dealing with dynamics of dianalytic transformations of nonorientable Klein surfaces. We are examining mainly the transformations of the real projective plane P^2, whose orientable double cover is the Riemann sphere $\overline{C}$. It is shown that the automorphisms of P^2 are projections of the rotations of $\overline{C}$ and some of the other dianalytic transformations of P^2 are projections of Blaschke products.

1 Introduction

It is known that the only analytic maps $F : \overline{C} -> \overline{C}$ are the rational functions. If $F(z) = p(z)/q(z)$, where p and q are polynomials with complex coefficients that have no common factors, then the degree of F is by definition, $\deg(F) = \max\{\deg(p), \deg(q)\}$, and it represents the number (counted with multiplicity) of inverse images of any point $z \in \overline{C}$ by the map F.

The Fatou-Julia theory applies to rational maps F whose degree is at least two. The dynamics of Mœbius transformations is in general simpler and in most cases can be completely described.

The use of computer graphics, as well as the application of quasiconformal mapping techniques has brought in the last decades an extraordinary development of Fatou-Julia theory. Moreover, parallel studies have been done on non rational analytic functions and extensions have been considered to Riemann surfaces, multivalued functions, several variables, etc. Our aim is to deal with dynamics on nonorientable Klein surfaces.

Such a study was initiated in [5], where a complete list of dianalytic self maps of the real projective plane, the pointed real projective plane and the Klein bottle has been given. The dynamics of these transformations can be described by using their lifting to the respective orientable double covers. In this paper we take a closer look to these dynamics.

Let us consider on the Riemann sphere the conformal structure (see [2]) induced by the atlas $\Upsilon = \{(C, \varphi_1), (\overline{C} - \{0\}, \varphi_2)\}$, where $\varphi_1(z) = z$, $\varphi_2(z) = 1/z$. The antianalytic involution $h : \overline{C} -> \overline{C}$ defined by $h(z) = -1/\overline{z}$ if $z \neq 0$, $z \neq \infty$, and $h(0) = \infty$, $h(\infty) = 0$ makes $(\overline{C}, h)$ into a symmetric Riemann surface. The real projective plane $P^2 = \overline{C} / < h >$

endowed with the unique dianalytic structure which makes the canonical projection $\pi : \overline{C} -> P^2$ a dianalytic function, represents a nonorientable Klein surface (see [3] and [4]).

2 Dianalytic Transformations of P^2

We have shown in [5] that any dianalytic transformation f of P^2 has the representation $f(\widetilde{z}) = \widetilde{F(z)}$, or $f(\widetilde{z}) = \widetilde{F(\overline{z})}$, where $\widetilde{z} = (z, h(z)\}$ and F is a rational function of the form:

(1) $F(z) = e^{i\theta} \frac{a_0 z^{2n+1} + a_1 z^{2n} + ... + a_{2n+1}}{-\overline{a}_{2n+1} z^{2n+1} + \overline{a}_{2n} z^{2n} - ... + \overline{a}_0}$, $|a_0| + |a_{2n+1}| \neq 0$, $\theta \in R$.

Also, any dianalytic automorphism of P^2 has the representation $g(\widetilde{z}) = \widetilde{G(z)}$, or $g(\widetilde{z}) = \widetilde{G(\overline{z})}$, where

(2) $G(z) = e^{i\theta} \frac{az+b}{-\overline{b}z+\overline{a}}$, $|a| + |b| \neq 0$, $\theta \in R$

By using these facts we can prove the following theorem.

Theorem 2.1 *Any dianalytic transformation* f *of* P^2 *has the representation* $f(\widetilde{z}) = \widetilde{F(z)}$ *or* $f(\widetilde{z}) = \widetilde{F(\overline{z})}$, *where*

(3) $F(z) = e^{i\alpha} \prod_{k=1}^{2n+1} \frac{z - z_k}{1 + \overline{z_k} z}$, $\alpha \in R$,

for some arbitrary , not necessarily distinctive

$z_k \in C$.

Proof. Indeed, at least one of the polynomials in (1) has the degree $2n+1$. Suppose that this is the polynomial at the numerator. Then $a_0 \neq 0$, and the polynomial has the decomposition $a_0(z - z_1)(z - z_2)...(z - z_{2n+1})$, where the numbers z_k are repeated, if they are multiple roots.

By making the change of variable $z = -1/\overline{w}$ at the denominator, this becomes:

$$\frac{\overline{a_{2n+1}}}{\overline{w}^{2n+1}} + \frac{\overline{a_{2n}}}{\overline{w}^{2n}} + ... + \overline{a_0} = \frac{1}{\overline{w}^{2n+1}} (\overline{a_0 w^{2n+1}} + \overline{a_1 w^{2n}} + ... + \overline{a_{2n+1}}) =$$
$$\frac{1}{\overline{w}^{2n+1}} \overline{(a_0 w^{2n+1} + a_1 w^{2n} + ... + a_{2n+1})} = \frac{1}{\overline{w}^{2n+1}} \overline{a_0 \prod_{k=1}^{2n+1} (w - z_k)} =$$
$$\overline{a_0} \prod_{k=1}^{2n+1} \frac{\overline{w - z_k}}{\overline{w}^{2n+1}} = \overline{a_0} \prod_{k=1}^{2n+1} \frac{\overline{w} - \overline{z_k}}{\overline{w}^{2n+1}} = \overline{a_0} \prod_{k=1}^{2n+1} (1 - \frac{\overline{z_k}}{\overline{w}}) = \overline{a_0} \prod_{k=1}^{2n+1} (1 + \overline{z_k} z)$$

Then F has the expression:

$$F(z) = e^{i\theta} \frac{a_0 \prod_{k=1}^{2n+1} (z - z_k)}{\overline{a_0} \prod_{k=1}^{2n+1} (1 + \overline{z_k} z)} = e^{i(\theta + 2\beta)} \prod_{k=1}^{2n+1} \frac{z - z_k}{1 + \overline{z_k} z} = e^{i\alpha} \prod_{k=1}^{2n+1} \frac{z - z_k}{1 + \overline{z_k} z}$$

where $\beta = \arg a_0$ and $\alpha = \theta + 2\beta$. ■

An analogous reasoning brings us to the same result if we suppose that the polynomial at the denominator has the degree $2n+1$.

3 Dynamics of Dianalytic Automorphisms of P^2

Theorem 1 has the following Corollary:

Corollary 3.1 *The dianalytic automorphisms of* P^2 *have the representation:*

$g(\widetilde{z}) = \widetilde{G(z)}$ *or* $g(\widetilde{z}) = \widetilde{G(\overline{z})}$, *where* G *is a rotation of the Riemann sphere.*

Proof. We need to show that $z -> e^{i\alpha} \frac{z - z_0}{1 + \overline{z_0} z}$ is indeed a rotation of the Riemann sphere. It is obvious that $w -> e^{i\alpha} w$ represents a rotation of the Riemann sphere around the vertical diameter. On the other hand, it can be easily checked that $z -> \frac{z - z_0}{1 + \overline{z_0} z}$ has the fixed points $ie^{i\theta}$ and $-ie^{i\theta}$, where $\theta = \arg z_0$, therefore this mapping represents a rotation of the Riemann sphere around the diameter passing through the two fixed points. The composition

of the two rotations give a new rotation of the Riemann sphere around a third diameter. The extremities of that diameter have as stereographic projections the fixed points of the transformation G. The dynamics of G is then described by the Steiner's net associated to those fixed points (see, for example [1], page 85). It can be easily shown that if $z_0 = re^{i\theta}$, while the rotation axis of $z-> \frac{z-z_0}{1+\overline{z}_0 z}$ depends on θ, the rotation angle is $\cos^{-1}\frac{r^2-1}{r^2+1}$. Therefore, whether the final rotation is rational or irrational depends on α as well as on r. In the case of rational rotations we have cyclic iterations. Due to the $h-$invariance of G, the order of the cycle should be even. Then g will have also cyclic iterations of an order half of that of G. ■

4 Dianalytic Maps by Blaschke Products

A Blaschke product is a function of the form

(4) $$F(z) = e^{i\theta}\prod\frac{z-z_k}{1-\overline{z}_k z}, \quad |z_k| < 1$$

with a finite or infinite number of terms. When there are infinitely many terms, the product is written usually under the form:

$$B(z) = \prod\frac{\overline{z}_k}{|z_k|}\frac{z_k-z}{1-\overline{z}_k z}$$

and it is known that if

(5) $$\sum(1-|z_k|) < \infty,$$

then it converges uniformly on compact subsets of $\overline{C}-E$, where $E \subseteq \partial D$ is the set of accumulation points of $\{z_k\}$ (here ∂D is the unit circle).

We notice that F is not generally $h-$invariant, therefore it is not of the form (3). However, if for every k there is k' such that $z_k = -z_{k'}$, then:

$$\frac{z-z_k}{1-\overline{z}_k z}\cdot\frac{z-z_{k'}}{1-\overline{z}_{k'} z} = \frac{z-z_k}{1-\overline{z}_{k'} z}\cdot\frac{z-z_{k'}}{1-\overline{z}_k z} = \frac{z-z_k}{1+\overline{z}_k z}\cdot\frac{z-z_{k'}}{1+\overline{z}_{k'} z}$$

is $h-$invariant. If such a condition is fulfilled and for an odd number of k, $z_k = 0$, and if the product is finite, then (4) is of the form (3) and the functions f defined by

$$\widetilde{z}->\widetilde{F(z)}, \text{ respectively } \widetilde{z}->\widetilde{F(\overline{z})}$$

are dianalytic transformations of P^2. Then

(6) $$F(z) = e^{i\theta}z^{2p+1}\prod_{k=1}^{m}\frac{z^2-z_k^2}{1-\overline{z}_k^2 z^2}, \;\; z_k \neq 0, \;\; k = 1, 2, ..., m$$

It is obvious that for such a function the unit disk remains invariant. We ignore the trivial case where $m = 0$ and suppose that $\deg(F) \geq 3$. By the Schwarz Lemma , for any z, with $|z| < 1$, we have $|F(z)| < |z|$. Consequently, the sequence $(|F^{\circ n}(z)|)$ is decreasing, which shows that the open unit disk D belongs to the Fatou set of F. Moreover, D is the basin of attraction of the attracting fixed point 0. By a known result, (see [6], or [8], p. 46), the Julia set of F must be the whole unit circle. Consequently, we have proved:

Theorem 4.1 *If $f : P^2 -> P^2$ is the projection of a finite Blaschke product F with $\deg(F) \geq 3$ then the Fatou set of f is $\{\widetilde{z} : |z| < 1\}$ and the Julia set of f is $\{\widetilde{z} : |z| = 1\}$.*

It follows that the dynamics of those dianalytic self-maps of P^2, which happen to be projections of finite Blaschke products, are very simple. In [8], p. 150, more general finite Blaschke products were considered, namely for which some of z_k are inside the unit circle and some are outside. It has been shown that such a product might possess Herman rings, therefore the structure of the Julia set and that of the Fatou set is much more complicated. For an $h-$invariant Blaschke product of that type, the respective Herman rings should be symmetric, such that they project in pairs on the same Herman ring in P^2.

Suppose now that the infinite Blaschke product $B(z)$ is $h-$invariant and the convergence condition (5) is fulfilled. This last condition implies that $B(z)$ is analytic in $\overline{C}$ with the exception the poles $1/\overline{z}_k$ and the points of E, which are essential singularities (see [1]). We have again that $|B(z)| < 1$ for every z with $|z| < 1$ and $|B(z)| > 1$ for every z with $|z| > 1$. Then, in this case too, the unit disk is completely invariant (see [6]), and so is the exterior of the unit disk.

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